ON THE MINIMAL MODEL THEORY FOR DLT PAIRS OF NUMERICAL LOG KODAIRA DIMENSION ZERO

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ABSTRACT. We prove the existence of good log minimal models for dlt pairs of numerical log Kodaira dimension 0.

CONTENTS

1. INTRODUCTION

Throughout this article, we work over $\mathbb{C}$, the complex number field. We will make use of the standard notation and definitions as in [KM] and [KMM]. The minimal model conjecture for smooth varieties is the following:

Conjecture 1.1 (Minimal model conjecture)**.** *Let X be a smooth projective variety. Then there exists a minimal model or a Mori fiber space of X.*

This conjecture is true in dimension 3 and 4 by Kawamata, Kollár, Mori, Shokurov and Reid (cf. [KMM], [KM] and [Sho2]). In the case where K_X is numerically equivalent to some effective divisor in dimension 5, this conjecture is proved by Birkar (cf. [Bi1]). When X is of general type or K_X is not pseudo-effective, Birkar, Cascini, Hacon

and McKernan prove Conjecture 1.1 in arbitrary dimension ([BCHM]). Moreover if X has maximal Albanese dimension, Conjecture 1.1 is true by [F2]. In this article, among other things, we consider Conjecture 1.1 in the case $\nu(K_X) = 0$ (for the definition of ν, see Definition 2.5):

Theorem 1.2. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair such that $\nu(K_X + \Delta) = 0$. Then there exists a minimal model (X_m, Δ_m) of (X, Δ).*

Actually the author heard from Vladimir Lazić that Theorem 1.2 for klt pairs has been proved by Druel (cf. [D]) after he finished this work. In this article, moreover, we give the generalization of his result for dlt pairs by using the sophisticated Birkar–Cascini–Hacon–McKernan's results and Druel's method. Essentially our method seems to be same as Druel's. However, by expanding this result to dlt pairs we give the different proof of the abundance theorem for log canonical pairs in the case $\nu = 0$ as shown by [CKP] and [Ka2]:

Theorem 1.3 (=Theorem 6.1)**.** *Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-divisor. Suppose that (X, Δ) is a log canonical pair such that $\nu(K_X + \Delta) = 0$. Then $K_X + \Delta$ is abundant, i.e. $\nu(K_X + \Delta) = \kappa(K_X + \Delta)$.*

For the first time, Nakayama proved Theorem 1.3 when (X, Δ) is klt. Nakayama's proof is independent of Simpson's results [Sim]. Simpson's results are used to approach the abundance conjecture in [CPT]. Campana–Perternell–Toma prove Theorem 1.3 when X is smooth and $\Delta = 0$. Siu also gave an analytic proof of it (cf. [Siu]). The results of [CKP], [Ka2] and [Siu] depend on [Sim] and [Bu]. In this article, we show Theorem 1.3 by using a method different from [CPT], [CKP], [Ka2] and [Siu]. In our proof of Theorem 1.3, we do not need results of [Sim] and [Bu]. Our proof depends on [BCHM] and [G].

We summarize the contents of this article. In Section 2, we define the *Kodaira dimension* and the *numerical Kodaira dimension*, and collect some properties of these. In Section 3, we review a *log minimal model program with scaling* and works of Birkar–Cascini–Hacon–McKernan. In Section 4, we introduce the *divisorial Zariski decomposition* and collect some properties of it. Section 5 is devoted to the proof of the existence of minimal models in the case $\nu = 0$. In Section 6, we prove Theorem 1.3.

Notation and Definition 1.4. Let $\mathbb{K}$ be the real number field $\mathbb{R}$ or the rational number field $\mathbb{Q}$. We set $\mathbb{K}_{>0} = \{x \in \mathbb{K} | x > 0\}$.

Let X be a normal variety and let Δ be an effective $\mathbb{K}$-divisor such that $K_X + \Delta$ is $\mathbb{K}$-Cartier. Then we can define the *discrepancy*

$a(E, X, \Delta) \in \mathbb{K}$ for every prime divisor E *over* X. If $a(E, X, \Delta) \geq -1$ (resp. > -1) for every E, then (X, Δ) is called *log canonical* (resp. *kawamata log terminal*). We sometimes abbreviate log canonical (resp. kawamata log terminal) to *lc* (resp. *klt*).

Assume that (X, Δ) is log canonical. If E is a prime divisor over X such that $a(E, X, \Delta) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X, Δ), where $c_X(E)$ is the closure of the image of E on X. For the basic properties of log canonical centers, see [F1, Section 9].

Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and D a $\mathbb{Z}$-Cartier divisor on X. We set the complete linear system $|D/S| = \{E | D \sim_{\mathbb{Z}, S} E \geq 0\}$ of D over S. The base locus of the linear system $|D/S|$ is denoted by $\mathrm{Bs}|D/S|$. When $S = \mathrm{Spec}\,\mathbb{C}$, we denote simply $|D|$ and $\mathrm{Bs}|D|$.

2. Preliminaries

Definition 2.1 (Classical Iitaka dimension, cf. [N, II, 3.2, Definition])**.** Let X be a normal projective variety and D an $\mathbb{R}$-Cartier divisor on X. If $|\llcorner mD \lrcorner| \neq \emptyset$, we put a dominant rational map

$$\phi_{|\llcorner mD \lrcorner|} : X \dashrightarrow W_m,$$

with respect to the complete linear system of $\llcorner mD \lrcorner$. We define the *Classical Iitaka dimension* $\kappa(D)$ of D as the following:

$$\kappa(D) = \max\{\dim W_m\}$$

if $H^0(X, \llcorner mD \lrcorner) \neq 0$ for some positive integer m and $\kappa(D) = -\infty$ otherwise.

Lemma 2.2. *Let Y be a normal projective variety, $\varphi : Y \to X$ a projective birational morphism onto a normal projective variety, and let D be an $\mathbb{R}$-Cartier divisor on X. Then it holds the following:*

(1) $\kappa(\varphi^* D) = \kappa(\varphi^* D + E)$ *for any φ-exceptional effective $\mathbb{R}$-divisor E, and*

(2) $\kappa(\varphi^* D) = \kappa(D)$.

Proof. (1) and (2) follows from [N, II, 3.11, Lemma]. $\square$

The following is remarked by Shokurov:

Remark 2.3. In general, $\kappa(D)$ may not coincide with $\kappa(D')$ if $D \sim_{\mathbb{R}} D'$. For example, let X be the $\mathbb{P}^1$, P and Q closed points in X such that $P \neq Q$ and a irrational number. Set $D = a(P - Q)$. Then $\kappa(D) = -\infty$ in spite of the fact that $D \sim_{\mathbb{R}} 0$.

However, fortunately, $\kappa(D)$ coincides with $\kappa(D')$ if D and D' are effective divisors such that $D \sim_{\mathbb{R}} D'$ ([Ch, Corollary 2.1.4]). Hence it seems reasonable that we define the following as the *Iitaka (Kodaira)* dimension for $\mathbb{R}$-divisors.

Definition 2.4 (Invariant Iitaka dimension, [Ch, Definition 2.2.1], cf. [CS, Section 7]). Let X be a normal projective variety and D an $\mathbb{R}$-Cartier divisor on X. We define the *invariant Iitaka dimension $\kappa(D)$* of D as the following:

$$K(D) = \kappa(D')$$

if there exists an effective divisor D' such that $D \sim_{\mathbb{R}} D'$ and $K(D) = -\infty$ otherwise. Let (X, Δ) be a log canonical. Then we call $K(K_X + \Delta)$ the *log Kodaira dimension* of (X, Δ).

Definition 2.5 (Numerical Iitaka dimension). Let X be a normal projective variety, D an $\mathbb{R}$-Cartier divisor and A an ample Cartier divisor on X. We set

$$\sigma(D, A) = \max\{k \in \mathbb{Z}_{\geq 0} | \limsup_{m \to \infty} m^{-k} \dim H^0(X, \llcorner mD \lrcorner + A) > 0\}$$

if $H^0(X, \llcorner mD \lrcorner + A) \neq 0$ for infinitely many $m \in \mathbb{N}$ and $\sigma(D, A) = -\infty$ otherwise. We define

$$\nu(D) = \max\{\sigma(D, A) | A \text{ is a ample divisor on } Y\}.$$

Let (X, Δ) be a log pair. Then we call $\nu(K_X + \Delta)$ the *numerical log Kodaira dimension* of (X, Δ). If $\Delta = 0$, we simply say $\nu(K_X)$ is the *numerical Kodaira dimension* of X.

Lemma 2.6. *Let Y be a normal projective variety, $\varphi : Y \to X$ a projective birational morphism onto a normal projective variety, and let D be an $\mathbb{R}$-Cartier divisor on X. Then it holds the following:*

(1) $\nu(\varphi^*D) = \nu(\varphi^*D + E)$ *for any φ-exceptional effective $\mathbb{R}$-divisor E,*

(2) $\nu(\varphi^*D) = \nu(D)$, *and*

(3) $\nu(D) = \max\{k \in \mathbb{Z}_{\geq 0} | D^k \not\equiv 0\}$ *when D is nef.*

Proof. See [N, V, 2.7, Proposition]. □

Lemma 2.7 ([N, V, 2.7, Proposition, (1)])**.** *Let X be a projective variety and D and D' $\mathbb{R}$-Cartier divisors on X such that $D \equiv D'$. Then $\nu(D) = \nu(D')$.*

Remark 2.8. $\nu(D)$ is denoted as $\kappa_\sigma(D)$ in [N, V, § 2]. Moreover Nakayama also defined $\kappa_\sigma^-(D)$, $\kappa_\sigma^+(D)$ and $\kappa_\nu(D)$ as some numerical Iitaka dimensions. In this article we mainly treat in the case where $\nu(D) = 0$, i.e. $\kappa_\sigma(D) = 0$. Then it holds that $\kappa_\sigma^-(D) = \kappa_\sigma^+(D) = \kappa_\nu(D) = 0$ (cf. [N, V, 2.7, Proposition (8)]). Moreover, if a log canonical pair (X, Δ) has a *weakly log canonical model* in the sense of Shokurov, then $\nu(K_X + \Delta)$ coincides with the *numerical log Kodaira dimension* in the sense of Shokurov by Lemma 2.6 (cf. [Sho1, 2.4, Proposition]).

Definition 2.9. Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and D an $\mathbb{R}$-Cartier divisor on X. We set

$$\mathbf{B}_\equiv(D/S) = \bigcap_{D \equiv_S E \geq 0} \text{Supp}\, E.$$

When $S = \text{Spec}\,\mathbb{C}$, we denote simply $\mathbf{B}_\equiv(D)$.

We introduce a *dlt blow-up*. The following theorem was originally proved by Professor Christopher Hacon (cf. [F1, Theorem 10.4], [KK, Theorem 3.1]). For a simpler proof, see [F3, Section 4]:

Theorem 2.10 (Dlt blow-up)**.** *Let X be a normal quasi-projective variety and Δ an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose that (X, Δ) is log canonical. Then there exists a projective birational morphism $\varphi : Y \to X$ from a normal quasi-projective variety with the following properties:*

(i) *Y is $\mathbb{Q}$-factorial,*

(ii) $a(E, X, \Delta) = -1$ *for every φ-exceptional divisor E on Y, and*
(iii) *for*

$$\Gamma = \varphi_*^{-1}\Delta + \sum_{E:\varphi\text{-}exceptional} E,$$

it holds that (Y, Γ) is dlt and $K_Y + \Gamma = \varphi^*(K_X + \Delta)$.

The above theorem is very useful for studying log canonical singularities (cf. [F1], [G], [KK]).

3. Log minimal model program with scaling

In this section, we review a *log minimal model program with scaling* and introduce works by Birkar–Cascini–Hacon–McKernan.

Lemma 3.1 (cf. [Bi1, Lemma 2.1] and [F1, Theorem 18.9])**.** *Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) a $\mathbb{Q}$-factorial projective log canonical pair such that Δ is a $\mathbb{K}$-divisor. Let H be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta + H$ is π-nef and $(X, \Delta + H)$ is log canonical. Suppose that $K_X + \Delta$ is not π-nef. We put*

$$\lambda = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_X + \Delta + \alpha H \text{ is } \pi\text{-}nef\}.$$

Then $\lambda \in \mathbb{K}_{>0}$ and there exists an extremal ray $R \subseteq \overline{NE}(X/S)$ such that $(K_X + \Delta).R < 0$ and $(K_X + \Delta + \lambda H).R = 0$.

Definition 3.2 (Log minimal model program with scaling)**.** Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) a $\mathbb{Q}$-factorial projective divisorial log terminal pair such that Δ is a $\mathbb{K}$-divisor. Let H be an effective $\mathbb{K}$-divisor such that $K_X + \Delta + H$ is π-nef and $(X, \Delta + H)$ is divisorial log terminal. We put

$$\lambda_1 = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_X + \Delta + \alpha H \text{ is } \pi\text{-nef}\}.$$

If $K_X + \Delta$ is not π-nef, then $\lambda_1 > 0$. By Lemma 3.1, there exists an extremal ray $R_1 \subseteq \overline{NE}(X/S)$ such that $(K_X + \Delta).R_1 < 0$ and $(K_X + \Delta + \lambda_1 H).R_1 = 0$. We consider an extremal contraction with respect to this R_1. If it is a divisorial contraction or a flipping contraction, let

$$(X, \Delta) \dashrightarrow (X_1, \Delta_1)$$

be the divisorial contraction or its flip. Since $K_{X_1} + \Delta_1 + \lambda_1 H_1$ is π-nef, we put

$$\lambda_2 = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_{X_1} + \Delta_1 + \alpha H_1 \text{ is } \pi\text{-nef}\},$$

where H_1 is the strict transform of H on X_1. Then we find an extremal ray R_2 by the same way as the above. We may repeat the process. We

call this program a *log minimal model program with scaling of H* over S. When this program runs as the following:

$$(X_0, \Delta_0) = (X, \Delta) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \cdots,$$

then

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots,$$

where $\lambda_i = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_{X_{i-1}} + \Delta_{i-1} + \alpha H_{i-1} \text{ is } \pi\text{-nef}\}$ and H_{i-1} is the strict transform of H on X_{i-1}.

The following theorems are slight generalizations of [BCHM, Corollary 1.4.1] and [BCHM, Corollary 1.4.2]. These seem to be well-known for the experts.

Theorem 3.3 (cf. [BCHM, Corollary 1.4.1])**.** *Let* $\pi : X \to S$ *be a projective morphism of normal quasi-projective varieties and* (X, Δ) *be a* $\mathbb{Q}$-*factorial projective divisorial log terminal pair such that* Δ *is an* $\mathbb{R}$-*divisor. Suppose that* $\varphi : X \to Y$ *is a flipping contraction of* (X, Δ). *Then there exists the log flip of* φ.

Proof. Since $-(K_X + \Delta)$ is φ-ample, so is $-(K_X + \Delta - \epsilon \llcorner \Delta \lrcorner)$ for a sufficiently small $\epsilon > 0$. Because $\rho(X/Y) = 1$, it holds that $K_X + \Delta \sim_{\mathbb{R}, Y} c(K_X + \Delta - \epsilon \llcorner \Delta \lrcorner)$ for some positive number c. By [BCHM, Corollary 1.4.1], there exists the log flip of $(X, \Delta - \epsilon \llcorner \Delta \lrcorner)$. This log flip is also the log flip of (X, Δ) since $K_X + \Delta \sim_{\mathbb{R}, Y} c(K_X + \Delta - \epsilon \llcorner \Delta \lrcorner)$. $\square$

Theorem 3.4 (cf. [BCHM, Corollary 1.4.2])**.** *Let* $\pi : X \to S$ *be a projective morphism of normal quasi-projective varieties and* (X, Δ) *be a* $\mathbb{Q}$-*factorial projective divisorial log terminal pair such that* Δ *is an* $\mathbb{R}$-*divisor. Suppose that there exists a* π-*ample* $\mathbb{R}$-*divisor* A *on* X *such that* $\Delta \geq A$. *Then any sequences of log minimal model program starting from* (X, Δ) *with scaling of* H *over* S *terminate, where* H *satisfies that* $(X, \Delta + H)$ *is divisorial log terminal and* $K_X + \Delta + H$ *is* π-*nef.*

The above theorem is proved by the same argument as the proof of [BCHM, Corollary1.4.2] because [BCHM, Theorem E] holds on the above setting.

4. Divisorial Zariski decomposition

In this section, we introduce the *divisorial Zariski decomposition* for a pseudo-effective divisor.

Definition 4.1 (cf. [N, III, 1.13, Definition] and [Ka1])**.** Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and D an $\mathbb{R}$-Cartier divisor. We call that D is a *limit of movable* $\mathbb{R}$-*divisors* over

S if $[D] \in \overline{\mathrm{Mov}}(X/S) \subseteq N^1(X/S)$ where $\overline{\mathrm{Mov}}(X/S)$ is the closure of the convex cone spanned by classes of fixed part free $\mathbb{Z}$-Cartier divisors over S. When $S = \mathrm{Spec}\,\mathbb{C}$, we denote simply $\overline{\mathrm{Mov}}(X)$.

Definition 4.2 (cf. [N, III, 1.6, Definition and 1.12, Definition]). Let X be a smooth projective variety and B a big $\mathbb{R}$ divisor. We define

$$\sigma_\Gamma(B) = \inf\{\mathrm{mult}_\Gamma B' | B \equiv B' \geq 0\}$$

for a prime divisor Γ. Let D be a pseudo-effective divisor. Then we define the following:

$$\sigma_\Gamma(D) = \lim_{\epsilon \to 0+} \sigma_\Gamma(D + \epsilon A)$$

for some ample divisor A. We remark that $\sigma_\Gamma(D)$ is independent of the choice of A. Moreover the above two definitions coincide for a big divisor because a function $\sigma_\Gamma(\cdot)$ on $\mathrm{Big}(X)$ is continuous where $\mathrm{Big}(X) := \{[B] \in \mathrm{N}^1(X) | B \text{ is big}\}$ (cf. [N, III, 1.7, Lemma]). We set

$$N(D) = \sum_{\Gamma:\text{prime divisor}} \sigma_\Gamma(D)\Gamma \text{ and } P(D) = D - N(D).$$

We remark that $N(D)$ is a finite sum. We call the decomposition $D = P(D) + N(D)$ the *divisorial Zariski decomposition* of D. We say that $P(D)$ (resp. $N(D)$) is the *positive part* (resp. *negative part*) of D.

Remark that the decomposition $D \equiv P(D) + N(D)$ is called several names: the *sectional decomposition* ([Ka1]), the *σ-decomposition* ([N]), the *divisorial Zariski decomposition* ([Bo]), and the *numerical Zariski decomposition* ([Ka2]).

Proposition 4.3. *Let X be a smooth projective variety and D a pseudo-effective $\mathbb{R}$-divisor on X. Then it holds the following:*

(1) *$\sigma_\Gamma(D) = \lim_{\epsilon \to 0+} \sigma_\Gamma(D + \epsilon E)$ for a pseudo-effective divisor E, and*

(2) *$\nu(D) = 0$ if and only if $D \equiv N(D)$.*

Proof. (1) follows from [N, III, 1.4, Lemma]. (2) follows from [N, V, 2.7, Proposition (8)]. $\square$

5. EXISTENCE OF MINIMAL MODELS IN THE CASE $\nu = 0$

Theorem 5.1 (cf. [D, Corollaire 3.4]). *Let X be a $\mathbb{Q}$-factorial projective variety and Δ an effective $\mathbb{R}$-divisor such that (X, Δ) is divisorial log terminal. Suppose that $\nu(K_X + \Delta) = 0$. Then any log minimal model programs starting from (X, Δ) with scaling of H terminate, where H satisfies that $H \geq A$ for some effective $\mathbb{R}$-ample divisor A, $(X, \Delta + H)$ is divisorial log terminal, and $K_X + \Delta + H$ is nef.*

Proof. Let $(X, \Delta) \dashrightarrow (X_1, \Delta_1)$ be a divisorial contraction or a log flip. Remark that it holds that

$$\nu(K_X + \Delta) = \nu(K_{X_1} + \Delta_1)$$

from Lemma 2.6 (1) and the negativity lemma. Now we run a log minimal model program

$$(X_i, \Delta_i) \dashrightarrow (X_{i+1}, \Delta_{i+1})$$

starting from $(X_0, \Delta_0) = (X, \Delta)$ with scaling of H. Assume by contradiction that this program does not terminate. Let $\{\lambda_i\}$ be as in Definition 3.2. We set

$$\lambda = \lim_{i \to \infty} \lambda_i.$$

If $\lambda \neq 0$, the sequence is composed by $(K_X + \Delta + \frac{1}{2}\lambda H)$-log minimal model program. Thus the sequence terminates by Theorem 3.4. Therefore we see that $\lambda = 0$. Now there exists j such that $(X_i, \Delta_i) \dashrightarrow (X_{i+1}, \Delta_{i+1})$ is a log flip for any $i \geq j$. Replace (X, Δ) by (X_j, Δ_j), we lose the fact that A is ample. Then we see the following:

Claim 5.2. $K_X + \Delta$ *is a limit of movable $\mathbb{R}$-divisors.*

Proof of Claim 5.2. See [Bi2, Step 2 of the proof of Theorem 1.5] or [F3, Theorem 2.3]. □

Let $\varphi : Y \to X$ be a log resolution of (X, Δ). We consider the divisorial Zariski decomposition

$$\varphi^*(K_X + \Delta) = P(\varphi^*(K_X + \Delta)) + N(\varphi^*(K_X + \Delta))$$

(Definition 4.2). Since

$$\nu(\varphi^*(K_X + \Delta)) = \nu(K_X + \Delta) = 0,$$

we see $P(\varphi^*(K_X + \Delta)) \equiv 0$ by Proposition 4.3 (2). Moreover we see the following claim:

Claim 5.3. $N(\varphi^*(K_X + \Delta))$ *is a φ-exceptional divisor.*

Proof of Claim 5.3. Let G be an ample divisor on X and ϵ a sufficiently small positive number. By Proposition 4.3 (1), it holds that

$$\operatorname{Supp} N(\varphi^*(K_X + \Delta)) \subseteq \operatorname{Supp} N(\varphi^*(K_X + \Delta + \epsilon G)).$$

If it holds that $\varphi_*(N(\varphi^*(K_X + \Delta))) \neq 0$, we see that $\mathbf{B}_\equiv(K_X + \Delta + \epsilon G)$ has codimension 1 components. This is a contradiction to Claim 5.2. Thus $N(\varphi^*(K_X + \Delta))$ is a φ-exceptional divisor. □

Hence $K_X + \Delta \equiv 0$, in particular, $K_X + \Delta$ is nef. This is a contradiction to the assumption. □

Corollary 5.4. *Let X be a $\mathbb{Q}$-factorial projective variety and Δ an effective $\mathbb{R}$-divisor such that (X, Δ) is divisorial log terminal. Suppose that $\nu(K_X + \Delta) = 0$. Then there exists a log minimal model of (X, Δ).*

Remark 5.5. These results are on the absolute setting. It may be difficult to extends these to the relative settings. See [F3].

6. ABUNDANCE THEOREM IN THE CASE $\nu = 0$

In this section, we prove the abundance theorem in the case where $\nu = 0$ for an $\mathbb{R}$-divisor:

Theorem 6.1. *Let X be a normal projective variety and Δ an effective $\mathbb{R}$-divisor. Suppose that (X, Δ) is a log canonical pair such that $\nu(K_X + \Delta) = 0$. Then $\nu(K_X + \Delta) = K(K_X + \Delta)$. Moreover, if Δ is a $\mathbb{Q}$-divisor, then $\nu(K_X + \Delta) = \kappa(K_X + \Delta) = K(K_X + \Delta)$.*

First we extends [G, Theorem 1.2] to an $\mathbb{R}$-divisor.

Lemma 6.2 (cf. [FG, Theorem 3.1]). *Let X be a normal projective variety and Δ an effective $\mathbb{K}$-divisor. Suppose that (X, Δ) is a log canonical pair such that $K_X + \Delta \equiv 0$. Then $K_X + \Delta \sim_{\mathbb{K}} 0$*

Proof. By taking a dlt blow-up (Theorem 2.10), we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. If $\mathbb{K} = \mathbb{Q}$, then the statement is nothing but [G, Theorem 1.2]. From now on, we assume that $\mathbb{K} = \mathbb{R}$. Let $\sum_i B_i$ be the irreducible decomposition of $\operatorname{Supp} \Delta$. We put $V = \bigoplus_i \mathbb{R} B_i$. Then it is well known that

$$\mathcal{L} = \{B \in V \mid (X, B) \text{ is log canonical}\}$$

is a rational polytope in V. We can also check that

$$\mathcal{N} = \{B \in \mathcal{L} \mid K_X + B \text{ is nef}\}$$

is a rational polytope and $\Delta \in \mathcal{N}$ (cf. [Bi2, Proposition 3.2] and [Sho1, 6.2 First Main theorem]). We note that $\mathcal{N}$ is known as Shokurov's polytope. Therefore, we can write

$$K_X + \Delta = \sum_{i=1}^{k} r_i (K_X + \Delta_i)$$

such that

    (i) Δ_i is an effective $\mathbb{Q}$-divisor such that $\Delta_i \in \mathcal{N}$ for every i,
    (ii) (X, Δ_i) is log canonical for every i, and
    (iii) $0 < r_i < 1$, $r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

Since $K_X + \Delta$ is numerically trivial and $K_X + \Delta_i$ is nef for every i, $K_X + \Delta_i$ is numerically trivial for every i. By [G, Theorem 1.2], we see that $K_X + \Delta \sim_{\mathbb{R}} 0$.

$\square$

Proof of Theorem 6.1. By taking a dlt blow-up (Theorem 2.10), we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. By Corollary 5.4, there exists a log minimal model (X_m, Δ_m) of (X, Δ). From Lemma 2.6 (3), it holds that $K_{X_m} + \Delta_m \equiv 0$. By Lemma 6.2, it holds that $K(K_{X_m} + \Delta_m) = 0$. Lemma 2.2 implies that $K(K_X + \Delta) = 0$. If Δ is a $\mathbb{Q}$-divisor, then there exists an effective $\mathbb{Q}$-divisor E such that $K_X + \Delta \sim_{\mathbb{Q}} E$ by Corollary 5.4 and Lemma 6.2. Thus we see that $\kappa(K_X + \Delta) = K(K_X + \Delta)$. We finish the proof of Theorem 6.1.

$\square$

Corollary 6.3. *Let $\pi : X \to S$ be a projective surjective morphism of normal quasi-projective varieties, and let (X, Δ) be a projective log canonical pair such that Δ is an effective $\mathbb{K}$-divisor. Suppose that $\nu(K_F + \Delta_F) = 0$ for a general fiber F, where $K_F + \Delta_F = (K_X + \Delta)|_F$. Then there exists an effective $\mathbb{K}$-divisor D such that $K_X + \Delta \sim_{\mathbb{K}, \pi} D$.*

Proof. This follows from Theorem 6.1 and [BCHM, Lemma 3.2.1]. $\square$

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